# Announcements

1) New Webwork up, due next Friday, supplement appearing later today

2) Extra credit opportunities - one due next Friday, the other next Monday

# Looking at Signals

( Section 4.8 )

Audio, Video, anything that is transmitted.

An audio signal in its purest form is continuous.

We will be interested in what happens when the signal is broken up into pieces (discrete)

Application: Discrete Time Signals (Audio)

(signal processing)

Let $S$ = the vector space of doubly infinite sequences of real numbers

$$(\ldots x_{-1}, x_0, x_1, x_2 \ldots)$$

Addition + scalar multiplication are coordinate-wise

## Example 1:

$$f(x) = \sin(x),$$

$x$ a real number

Can make a discrete signal by letting

$$x_k = \begin{cases} 0, & k = 0 \\ \sin\left(\frac{2\pi}{k}\right), & k \neq 0 \end{cases}$$

$$\ldots, -1, -\frac{\sqrt{3}}{2}, 0, 0, 0, 0, 0, \frac{\sqrt{3}}{2}, 1, \ldots$$

$$\ldots x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, x_4, \ldots$$

**Convention:** (starting point)

If there is a definite starting point for your signal, call this $x_0$ and suppress any negative indices:

$$(x_0, x_1, x_2, x_3, \ldots)$$

# CD vs. MP3

## Fight!

CD : 44,100 sampes/s (Hz)

MP3 : 8,000 – 48,000 Hz

Human perceptibility : 20 - 20,000 Hz

**Problem:** How do you separate signals?

Linear independence!

**Casorati Matrix:** To simplify, just take 3 signals.

Let $(x_k)$, $(y_k)$, and $(z_k)$ be signals. We want to know when these are linearly independent in the space of signals.

This means if there

are numbers $c_1, c_2, c_3$

with

$$c_1 (x_k) + c_2 (y_k) + c_3 (z_k) = (0)$$

sequence with zeroes in every entry

then $c_1 = c_2 = c_3 = 0$.

We can reduce to

$$c_1 x_k + c_2 y_k + c_3 z_k = 0$$

for all $k$.

Interpret this using matrices: Take a 3x3 matrix

$$\underbrace{\begin{bmatrix} x_k & y_k & z_k \\ x_{k+1} & y_{k+1} & z_{k+1} \\ x_{k+2} & y_{k+2} & z_{k+2} \end{bmatrix}}_{\text{Casorati Matrix } C_k} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If there is a value of $k$ for which $C_k$ is invertible, then $c_1 = c_2 = c_3 = 0$.

## Example 2:

$$x_k = (-1)^k$$
$$y_k = 2^k$$
$$z_k = 3^k$$

Are these signals linearly independent?

Construct Casorati Matrix

$$\begin{bmatrix} (-1)^k & 2^k & 3^k \\ (-1)^{k+1} & 2^{k+1} & 3^{k+1} \\ (-1)^{k+2} & 2^{k+2} & 3^{k+2} \end{bmatrix}$$

Let's plug in some values of k and check whether we get invertibility.

$k=0$

$$C_0 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$

$\det(C_0) = 12 \neq 0$,

so $C_0$ is invertible. This gives us $(x_k), (y_k), (z_k)$ are linearly independent.

Q: How to make this efficient?

A: Under certain conditions....

given later!

# Linear Difference Equations

Given real numbers

$a_0, a_1, \cdots, a_n$ and

signals $(y_k)$ and $(z_k)$,

a linear difference equation

is an equation of the form

$$\boxed{a_0 y_{k+n} + a_1 y_{k+n-1} + \cdots + a_{n-1} y_{k+1} + a_n y_k = z_k}$$

for all $k$.

$z_k = 0$ for all $k$: homogeneous

$z_k \neq 0$ for some $k$: not homogeneous

**Subspaces** The set of all solutions $(y_k)$ to a given homogeneous linear difference equation is a subspace of the space of signals.

From the book:

If three sequences satisfy the same homogeneous linear difference equation, then either the Casorati matrix is invertible for all k or is not invertible for any k. First case is linear independence, second is linear dependence.

**Problem:** Given a linear difference equation, can you find a solution?

**Homogeneous** at least the zero vector (0) is a solution since the set of solutions is a subspace.

**Result:** If $a_n \neq 0$, then

$$\sum_{j=0}^{n} a_{n-j}\, y_{k+j} = z_k$$

has a unique solution if $\{y_0, y_1, \ldots, y_{n-1}\}$ are given.

**Note:** In the homogeneous case, we get: if $a_n \neq 0$, then the set of solutions to

$$\sum_{j=0}^{n} a_{n-j} y_{k+j} = 0$$

is $n$ dimensional.

Example 3:

$$x_k = (-1)^k$$
$$w_k = 2^k$$
$$z_k = 3^k$$

$$y_{k+3} - 4y_{k+2} + y_{k+1} + 6y_k = 0$$

Check that $(x_k)$, $(w_k)$, and $(z_k)$ satisfy the equation. Do they form a basis for the set of solutions?

Our coefficients are $(1, -4, 1, 6)$,

which gives that the subspace of solutions is $4-1=3$ dimensional.

We already know $(x_k)$, $(w_k)$ and $(z_k)$ are linearly independent. Therefore, they are a basis!

Now let's check that the sequences satisfy the equation

$$y_{k+3} - 4y_{k+2} + y_{k+1} + 6y_k = 0$$

If $y_k = 2^k$,

$$2^{k+3} - 4\,2^{k+2} + 2^{k+1} + 6\,2^k$$

$$= 2^k(8 - 16 + 2 + 6)$$

$$= 0 \checkmark$$

If $y = 3^k$,

$$3^{k+3} - 4 \cdot 3^{k+2} + 3^{k+1} + 6 \cdot 3^k$$

$$= 3^k (27 - 36 + 3 + 6)$$

$$= 0 \checkmark$$

Finally, if $y = (-1)^k$

$$(-1)^{k+3} - 4 \cdot (-1)^{k+2} + (-1)^{k+1} + 6 \cdot (-1)^k$$

$$= (-1)^k (-1 - 4 - 1 + 6) = 0 \checkmark$$